NOTES ON GAUSS' THEORIA MOTUS.

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WHILE at Harvard College Observatory in 1858, I began reading Davis' translation of the Theoria Motus. Since my mathematical knowledge had been got wholly from the small text books of our schools, the difficulty of reading such a work as this, and of obtaining a clear idea of the processes used by the author, was very great. I well remember that at first the style of the work fairly took me off my feet, and seemed to leave me dangling in the air for a month or two before I could get a firm foot-hold; and how at last the beauty and power of Gauss' methods were seen and felt. Having no teacher nor any one to assist me, I made it a rule to work out every equation and all the numerical examples before going on; and in this slow way I read to article 164. The rest of the book was passed over with less care. The whole reading occupied me nearly a year; and I consider myself very fortunate in having almost by chance hit upon one of the most perfect books ever written on theoretical astronomy. Admiral Davis' translation was sharply criticised in the American Journal of Science, Volume 26, p. 147, and some mistakes were pointed out. There are in fact a few errors of translation that confuse the meaning, but on the whole the translation seems to me well done, and the errors in the formulæ are very few.

I wish to notice here a few of the points and reductions that gave me the most trouble; and I have found since that these are also the ones over which students are apt to stumble.

(1). In article 3 Gauss uses a linear form for the equation of conic sections

$$r + \alpha x + \beta y = \gamma,$$

in which γ is always positive; and he calls this the characteristic equation of these curves. We see at once that this is an equation of the second degree, and hence the reader pays but little attention to it; but by referring to the Mécanique Celeste, Tome I, p. 165, the correctness and meaning of the statement will be seen.

(2). Among the formulæ of article 8, the following seem to me better for computing the true anomaly and the radius vector than those that Gauss recommends,

$$r \sin v = a \cos \varphi \sin E,$$

$$r \cos v = a(\cos E - e).$$

(3). The indirect method given in article 11 for the solution of Kepler's problem is probably the best way of solving this famous question that has ever been given in the case of orbits where the eccentricity is not very small. The opportunity that is left for the computer to use his judgement and make a close guess is a great advantage.

(4). Since the coefficient of $d\varphi$ in article 16 is generally found with difficulty I give my reductions. Substituting the values of $\frac{dp}{p}$ and $\frac{e \sin v}{1 + e \cos v}$ $\times dv$, the value of dv being taken from the preceding article, we have

$$\frac{dr}{r} = \frac{da}{a} + \frac{a}{r}.\tan\varphi \sin v . dM + \left\{ \frac{(2 + e \cos v) e \sin v^2}{\cos\varphi(1 + e\cos v)} - \frac{\cos\varphi\cos v}{1 + e\cos v} - 2\tan\varphi \right\} d\varphi$$

By successive reductions and noticing that $e = \sin\varphi$, the coefficient of $d\varphi$ takes the forms

$$+\frac{2\sin\varphi \sin v^2 + \cos v \sin\varphi^2 \sin v^2 - 2\cos v \sin\varphi^2 - \cos v \cos\varphi^2 - 2\sin\varphi}{\cos\varphi(1 + e\cos v)},$$

$$-\frac{2\sin\varphi\cos v^2 + \sin\varphi^2\cos v^3 + \cos v \sin\varphi^2 + \cos v \cos\varphi^2}{\cos\varphi(1 + e\cos v)},$$

$$-\frac{(1 + e\cos v)^2.\cos v}{\cos\varphi(1 + e\cos v)}.$$

Hence since

$$\frac{a}{r} = \frac{1 + e\cos v}{\cos\varphi^2},$$

we have

$$dr = \frac{r}{a}.da + a\tan\varphi\sin v . dM - a\cos\varphi\cos v . d\varphi.$$

On sending the above reduction to Dr. Brünnow, then Prof'r of astronomy in the University of Michigan, in return I received from him the following solution.

All the equations between E, v and φ can be found from the formulæ of spherical trigonometry by means of a triangle given by B. Nicolai, a student with Gauss at Göttingen, and afterwards Director of the Observatory at Mannheim. From the equations given in article 8 we may assume the spherical triangle whose angles are $90° + E$; $90° - v$; and φ; the sides opposite being $90° - E$; $90° - v$, and φ. Thus we have

$$\sin E = \frac{\cos\varphi\sin v}{1 + e\cos v}; \qquad \cos E = \frac{\cos v + e}{1 + e\cos v};$$

$$\sin v = \frac{\cos\varphi\sin E}{1 - e\cos E}; \qquad \cos v = \frac{\cos E - e}{1 - e\cos E}.$$

From these equations we find

$$\cos E \cos v = \frac{\cos v^2 + e\cos v}{1 + e\cos v}; \quad \sin E \cos v = \frac{\cos\varphi \sin v \cos v}{1 + e\cos v};$$

$$\sin v \cos E = \frac{\cos\varphi \sin E \cos E}{1 - e\cos E}; \quad \cos v \sin E = \frac{\sin E\cos E - e\sin E}{1 - e\cos E}.$$

If we multiply the equation

$$\cos E \sin v = \frac{\cos v \sin v + e\sin v}{1 + e\cos v},$$

by $\cos\varphi$, and subtract the product from the value of $\sin E \cos v$ we have

$$\sin E\cos v - \cos E \sin v \cos\varphi = \frac{-e\cos\varphi \sin v}{1 + e\cos v} = -e\sin E.$$

Since $e = \sin\varphi$,

$$\cos E\cos v\cos\varphi = \frac{\cos\varphi - \cos\varphi\sin v^2 + \sin\varphi\cos\varphi\cos v}{1 + e\cos v}$$

$$= \cos\varphi - \sin E\sin v.$$

Hence

$$\cos\varphi = \sin v\sin E + \cos v\cos E\cos\varphi,$$

$$\sin\varphi\sin E = -\cos v\sin E + \sin v\cos E\cos\varphi,$$

$$\sin\varphi\cos E = \sin\varphi\cos E.$$

Similarly

$$\cos\varphi = \sin v\sin E + \cos v\cos E\cos\varphi,$$

$$\sin\varphi\sin v = \sin v\cos E - \cos v\sin E\cos\varphi,$$

$$\sin\varphi\cos v = \cos v\sin\varphi;$$

and these are the fundamental equations of spherical trigonometry.

If now we differentiate the equation

$$r = a(1 - e\cos E)$$

and then substitute the value of dE we find

$$dr = \frac{r}{a}.da + a\tan\varphi\sin v.dM + a.\left\{\sin\varphi\sin v\sin E - \cos\varphi\cos E\right\}.d\varphi.$$

Nicolai's triangle gives

$$\cos\varphi = \sin v\sin E + \cos v\cos E\cos\varphi,$$

$$\sin\varphi\sin v = \sin v\cos E - \cos v\sin E\cos\varphi.$$

Hence the coefficient of $ad\varphi$ in the value of dr is $-\cos\varphi\cos v$.

(5) In computing the maximum error for the hyperbola in section VII of article 32, I was obliged to solve a complete equation of the fourth degree. The numerical results agreed with those given by Gauss, but perhaps they may be got more easily, since in the other cases they are found by quadratic equations.

(6). In article 54 we have the elegant equations used by Gauss for the solution of a spherical triangle when from one side and the adjacent angles the other parts are to be found. These formulæ were published by Delambre in 1807 (Conn. des Tems, 1809, p. 445); and formulæ resembling them were given by Professor Mollweide in Germany; but it does not appear that Delambre or any one made use of these formulæ until they were applied by Gauss, and their convenience had been pointed out by him. In his review of the Theoria Mot. (Conn. des Tems 1812) Delambre says: "Quand j'eus trouve ces formules j'en cherchai les applications qui pouvaient etre vraiment utile; n'en voyant aucune, je les donnai simplement comme curieuses." When we remember that Gauss had used these formulæ several years before their publication by Delambre, and that it was through Gauss' example that they came into general use, it seems to be only fair that they should be called the Gaussian formulæ, although of late the French, and some English writers, call them Delambre's formulæ. These formulæ are frequently used when it would be better and more accurate, I think, to apply the three fundamental equations of spherical trigonometry, with addition and subtraction logarithms.

(7) The method given in articles (55) and (56) for computing the equatorial coordinates of a planet is simple. In the appendix Admiral Davis has added other equations given by Gauss in his first paper on this method, and among these the convenient equation for checking the calculation, into which it will be noticed all the auxiliary quantities enter:

$$\tan i = \frac{\sin b \sin c \sin (C - B)}{\sin a \cos A}.$$

By an error of print the denominator is given as $\sin a \sin A$. This method of computing by rectangular coordinates is objected to by Hansen who prefers polar coordinates as being more accurate. Differential formulæ can be found for changing the Gaussian auxiliaries from one equinox to another, but they do not seem to give any practical advantage.

(8). The continued fractions given in articles 90 and 100 are explained by Gauss in one of his memoirs. [See Runkle's Mathematical Monthly, Vol. III, p. 262.]

(9). A very elegant and satisfactory discussion of Lambert's equation for each of the conic sections will be fonnd in articles 106 to 110. Many of the discussions of this problem given in recent books on astronomy and rational mechanics are not much more than Gauss' discussion spoiled. In section IV will be found a condensed but pretty complete statement of the most useful relations between several places in space, such as that used by Olbers in computing the orbit of a comet.

(10). I will give one more note, and this on the derivation of the check equations at the end of article 140; the last of which gave me as much trouble as any thing in the whole book notwithstanding Gauss says: "quarum tamen derivationem non ita difficilem brevitatis caussa supprimimus."

If in figure 4 we extend the great circle $B''B$ to meet the circle AA'' in N', and apply equation I, article 140, we shall have

$$0 = \sin(l''-l')\sin N'A - \sin(l''-l)\sin N'A' + \sin(l'-l)\sin N'A''. \quad (1)$$

Proceeding now as in the beginning of article 140, and dividing equation (1) by R'' we have the first of the check equations. For the second equation, transpose and get the value of b from the first, and if we notice that

$$\sin(A''D'-\delta'') = \frac{\sin B \sin BB''}{\sin \varepsilon'}; \quad \sin(AD'-\delta) = \frac{\sin B'' \sin BB''}{\sin \varepsilon'}:$$

we shall have

$$b = \frac{R' \sin \delta'}{R'' \sin \delta''} \cdot \frac{\sin BB''}{\sin(\delta'-\sigma)\sin(AD'-\delta)\sin \varepsilon'} \left[\frac{\text{s.}\,\delta\,\text{s.}\,(l''-l)\,\text{s.}\,B + \text{s.}\,\delta''\,\text{s.}\,(l'-l)\,\text{s.}\,B''}{\sin(l''-l)}\right]$$

But we have

$$\sin N'A \sin N' = \sin \delta \sin B: \quad \sin N'A'' \sin N' = \sin \delta'' \sin B'',$$

and by (1) the factor in the brackets is reduced to

$$\sin N'A' \sin N'.$$

From the last equation of article 110 we have

$$\sin BB'' = \frac{\sin(\alpha''-\alpha)\cos\beta\cos\beta''}{\cos N'},$$

and the value of b becomes

$$b = \frac{R' \sin \delta'}{R'' \sin \delta''} \cdot \frac{\cos\beta\cos\beta''}{\sin(\delta'-\sigma)\sin(AD'-\delta)\sin\varepsilon'} \cdot \sin(\alpha''-\alpha)\sin N'A' \tan N'.$$

It is required therefore to reduce the factor $\sin(\alpha''-\alpha)\sin N'A'\tan N'$ to the form

$$\tan\beta\sin(\alpha''-l') - \tan\beta''\sin(\alpha-l') = S.$$

For this purpose I put

$$a'' - a = (a'' - l') - (a - l')$$

and expand $\sin(a'' - a)$. From the points B and B'' let fall perpendiculars on the great circle AA'', and denote the points of intersection by a and a''. We have

$$N'A' = N'a - (a - l'):\ N'A' = N'a'' - (a'' - l'), \qquad (2)$$

$$\tan N' = \frac{\tan\beta}{\sin N'a} = \frac{\tan\beta''}{\sin N'a''}.$$

By means of these equations the factor $\sin(a'' - a)\sin N'A' \tan N'$ takes the symmetrical form

$$+\sin(a''-l')\tan\beta\cos^2(a-l') - \sin(a''-l')\tan\beta\cos(a-l')\sin(a-l')\cot N'a$$
$$-\sin(a-l')\tan\beta''\cos^2(a''-l') + \sin(a-l')\tan\beta''\sin(a''-l')\cos(a''-l')\cot N'a''.$$

If now we change $\cos^2$ to $1 - \sin^2$, we have the value of S and the remaining terms reduce to

$$\sin(a''-l')\sin(a-l')\tan N'.\left\{\cos[N'a''-(a''-l')] - \cos[N'a-(a-l')]\right\} = 0,$$

by (2). Hence we have the value given by Gauss

$$b = \frac{R'\sin\delta'}{R''\sin\delta''}\cdot\frac{\cos\beta\cos\beta''.S}{\sin(\delta'-\sigma)\sin(AD'-\delta)\sin\varepsilon'}.$$

The preceding reduction is long, and quite likely it may be simplified. It will be seen that the relation

$$\sin(a''-a)\sin N'A'\tan N' = \tan\beta\sin(a''-l') - \tan\beta''\sin(a-l')$$

may be stated as a general theorem in spherical trigonometry, and perhaps some one may give a proof of it.

In reading a book like the Theoria Motus, the student is apt to be discouraged by the feeling that the writer of such a book was a man of wonderful power, and that his own efforts can not have much value. Now while such men were no doubt gifted with great natural ability, yet the history of their lives and their manner of work is full of encouragement to every earnest student. Thus we know that the first method devised by Gauss for computing the orbit of a planet was rude and clumsy compared with the elegant form in which it was published. It was by keeping the the problem steadily before his mind for several years, and carefully working out all its parts, that Gauss brought his solution at last to a form almost perfect. In the case of Laplace we know that at first he had erroneous notions on several subjects, and made mistakes, but he had the good sense and perseverence to correct his own errors, and at last produced the *Me´canique Celeste.*